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# Anomalous Magnetic Moment of Anyons in three dimensional $CP^{N-1}$ Model

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**Abstract**

We calculate the anomalous magnetic moment of anyons in three dimensional $CP^{N-1}$ model with a Chern-Simons term in various limits in $1/N$ expansion. We have found that for anyons of infinite mass the gyromagnetic ratio ($g$-factor) is 2 up to the next-to-leading order in $1/N$. Our result supports a recent claim that the $g$-factor of nonrelativistic anyons is exactly two. We also found that for $-\frac{8}{\pi} < \theta < 0$, the electromagnetic interation between two identical aynons of large mass are attractive.

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One of the peculiarities of planar physics is that there exists a particle of arbitrary spin, called anyon, since the rotation group in the two-dimensional space is $U(1)$. In field theories, anyon can be realized by minimally coupling a boson or a fermion to Chern-Simons gauge fields[1]. Due to the coupling to the Chern-Simons gauge fields, the wave function of two identical particles gets an Aharonov-Bohm phase, $\frac{4\pi}{\kappa}$, where $\kappa$ is the coefficient of the Chern-Simons term in the Lagrangian. Since the Aharonov-Bohm phase is interpreted as a statistical phase, the particle gets an induced spin, $S = 1/\kappa$. Anyon has been studied extensively because of its possible application to some of the planar condensed matter systems. It is found that anyon matter exhibits superconductivity[2] and describes quantum Hall system[3].

To study the many-body effects of anyons, one needs to find out how they interact with each other. Besides the statistical interaction due to the Chern-Simons fields, anyons may interact electro-magnetically. Especially, the interaction between its magnetic moment and electric charge can be very important since it may lead to a bound state of anyon and a magnetic flux tube[4], which is introduced by Jain[5] to describe fractional quantum Hall effect. Depending on the sign of the magnetic moment of anyon, the interaction of identical anyons could be attractive while the electric charge-charge interaction is always repulsive. Therefore, for a certain value of the coefficient of the Chern-Simons term, two identical anyons can form a bound state. Finding out the exact value of the magnetic moment of anyon is therefore of some importance.

There have been several calculations on the magnetic moment of anyon. Chou *et al.* showed that the $g$-factor of nonrelativistic anyonic particles is exactly two [6]. Also, in field theoretical models, it is shown that in the pure anyonic limit (namely for large value of the coefficient of the Chern-Simons term) the gyromagnetic ratio of anyon at one-loop is $g = 2 + O\left(\frac{1}{\kappa}\right)$ where $\kappa$ is the coefficient of the Chern-Simons

term[7, 8].

In this letter, we calculate the magnetic moment of anyons in the (2+1)-dimensional $CP^{N-1}$ model with a Chern-Simons term[9, 10] and we try to see if we reproduce the Chou *et al.*'s result for anyons of large mass. For $N = 2$, the model was originally studied to describe the high $T_c$ superconductivity [11, 12], since the model appears as a long-wavelength limit of the Hubbard model near half-filling[12].

The $CP^{N-1}$ model with a Chern-Simons term is described by a Lagrangian density (in the Euclidean notation)

$$\mathcal{L}_{CP^{N-1}} = |D_\mu n_i|^2 + \alpha \left( |n_i|^2 - \frac{N}{g^2} \right) + iN\frac{\theta}{16}\epsilon_{\mu\nu\lambda}A_\mu \partial_\nu A_\lambda, \tag{1}$$

where $i = 1, \cdots, N$ and the covariant derivative $D_\mu = \partial_\mu + iA_\mu$. In the original $CP^{N-1}$ model without the Chern-Simons term, the vector field $A_\mu$ is an auxiliarly field satisfying the constraint $A_\mu = (i/2)\bar{n}_i \overleftrightarrow{\partial}_\mu n_i$ which reduces the independent components of the complex fields $n_i$ by 2 together with the constraint on the "length square" of $n_i$, $|n_i|^2 = \frac{N}{g^2}$. But, as is well known, the quantum corrections generate a kinetic term for $A_\mu$. With the Chern-Simons term the $CP^{N-1}$ model behaves quite differently from the original one. The added Chern-Simons term in the lagrangian (1) not only makes the vector field $A_\mu$ propagte even at the classical level but also provides an induced fractional spin to $n_i$ by the Aharonov-Bohm effect [13].

In the $1/N$ perturbation, the model exhibits a nontrivial phase structure[14, 9, 10]. For $\frac{1}{g^2} < \frac{1}{g_c^2} \equiv \int [d^3p/(2\pi)^3](1/p^2)$, $\langle\alpha\rangle \neq 0$ and $\langle n_i \rangle = 0$. Therefore, the $n$ fields get mass, $M = \sqrt{\langle\alpha\rangle}$, and the flavor symmetry $SU(N)$ is unbroken. For $\frac{1}{g^2} > \frac{1}{g_c^2}$, the flavor symmetry breaks down to $U(N-1)$, leaving $2N-2$ massless Nambu-Goldstone bosons.

We calculate the anomalous magnetic moment of anyons described by (1) in the symmetric phase. It is well known that the magnetic moment of a particle can be infered from the on-shell matrix element of the effective interaction Hamiltonian

under a (spatially) uniform external magnetic field;

$$\langle p'|\mathcal{H}_{\rm int}|p\rangle = -\int d^2x A_\mu(x)\, \langle p'|j^\mu(x)|p\rangle\,, \tag{2}$$

where $j_\mu = i\bar{n}_i \overleftrightarrow{\partial}_\mu n_i$. The matrix element of the current of a scalar field is given by

$$\langle p'|j^\mu(x)|p\rangle = P^\mu F_1(q^2) - \frac{i}{2M}\epsilon^{\mu\nu\lambda} q_\nu P_\lambda F_2(q^2), \tag{3}$$

where $P^\mu = p'^\mu + p^\mu$ and $q^\mu = p'^\mu - p^\mu$. $F_1(q^2)$ corresponds to the electric charge form factor. For a constant magnetic field, $B$, one can easily see that, as $p' \to p$,

$$\langle p'|\mathcal{H}_{\rm int}|p\rangle = -\frac{B}{2M} F_2(0). \tag{4}$$

Therefore $\mu = F_2(0)/2M$ is the magnetic moment of a scalar field which couples minimally to photon. There are seven diagrams, shown in fig.1, which are leading order corrections in $1/N$ to the vertex function. By Lorentz invariance, the diagram (b) must be proportional to $q_\mu$, which is then zero due to the current conservation. The diagrams (c) and (d) are zero by explicit calculations. The diagram (e) in fig.1 vanishes identically, because of the symmetry of the Lagrangian under $n^i \to \bar{n}^i$, $A_\mu \to -A_\mu$, which is similar to Furry's theorem. Finally, one can easily see that the parity-odd parts of the diagrams (f) and (g) are zero by the similar argument used to prove the Coleman-Hill theorem on non-renormalization of the Chern-Simons term [15]. The diagram (f) gives

$$\Gamma^{(\rm f)}_\mu = \int \frac{d^3k}{(2\pi)^3} G_{\alpha\beta}(k) \frac{(2p-k)_\alpha}{(p-k)^2 + M^2} \Sigma(q-k) \Gamma_{\beta\mu}(k, q-k, -q), \tag{5}$$

where $G_{\alpha\beta}(k)$ is the propagator for the gauge fields and $\Sigma(q-k)$ is the propagator for $\alpha$ field. $\Gamma_{\beta\mu}(k, q-k, -q)$ is the one-loop effective $AA\alpha$ vertex. Because of the current conservation and the analyticity of the vertex, $\Gamma_{\beta\mu}(k, q-k, -q) = k_\beta G_\mu(k,q)$, where $G_\mu(k,q)$ is analytic at the zero momenta. (Note that the effecive vertex is analytic at the vanishing external momenta, since the $n$ fields are massive.) Since in

the eq. (5) $\Gamma_{\beta\mu}(k, q-k, -q)$ is contracted with the gauge field propagator $G_{\alpha\beta}(k)$, only the parity-even part of the propagator survives in the eq. (5). Therefore the only diagram which has a nonvanishing parity-odd part is diagram (a). Since the magnetic moment is parity-odd, it is the only diagram that contributes to the magnetic moment of anyon in the leading order in $1/N$. We calculate the diagram (a) under the condition that both initial and final $n$ fields are on the mass-shell;

$$\Gamma_\mu = \int \frac{d^3k}{(2\pi)^3} \frac{(2p-k)_\lambda (2p-2q-k)_\nu (2p-2k-q)_\mu}{[(p-k)^2 - M^2]\,[(p-k-q)^2 - M^2]} G_{\lambda\nu}. \tag{6}$$

We follow the Feynman rules for the $CP^{N-1}$ model derived in the reference [9]. To utilize the Feynman parametrization in evaluating the vertex function (6) we rewrite the gauge field propagator in the Källen-Lehman representation, in which the propagator is given by

$$G_{\mu\nu}(p) = \int_0^\infty ds \frac{\rho_1(s)}{p^2 - s + i\epsilon} \left( \delta_{\mu\nu} - \frac{p_\mu p_\nu}{p^2} \right) + \frac{\theta}{16} \int_0^\infty ds \frac{\rho_2(s)}{p^2 - s + i\epsilon} \epsilon_{\mu\rho\nu} p^\rho, \tag{7}$$

where

$$\rho_1(s) = \frac{i}{\pi} \frac{\Gamma(s)}{\Gamma^2(s) + \left(\frac{\theta}{16}\right)^2 s}, \quad \rho_2(s) = \frac{i}{\pi} \frac{1}{\Gamma^2(s) + \left(\frac{\theta}{16}\right)^2 s} \tag{8}$$

with

$$\Gamma(s) = \frac{1}{2}(s + 4M^2) \frac{\tan^{-1} \frac{\sqrt{s}}{2M}}{4\pi\sqrt{s}} - \frac{M}{4\pi}. \tag{9}$$

The parity-odd part of the vertex function which contributes to the magnetic moment is

$$\Gamma_\mu^{\rm odd} = \frac{\theta}{16} \int_0^\infty ds \rho_2(s) \int \frac{d^3k}{(2\pi)^3} \frac{(2p-k)_\lambda (2p-2q-k)_\nu (2p-2k-q)_\mu \epsilon_{\lambda\rho\nu} k^\rho}{[(p-k)^2 - M^2]\,[(p-k-q)^2 - M^2]\,[k^2 - s + i\epsilon]}. \tag{10}$$

After some calculations, we get

$$\Gamma_\mu^{\rm odd} = -\frac{i\theta}{32\pi} \epsilon_{\lambda\nu\mu} p_\lambda q_\nu \int_0^\infty ds \rho_2(s) F(M^2, s, q^2), \tag{11}$$

where

$$F(M^2, s, q^2) = \int_0^1 dx \int_0^x dy \left[M^2x^2 + (s - i\epsilon)(1-x) - q^2y(x-y)\right]^{-1/2}. \tag{12}$$

The magnetic form factor is then

$$F_2(q^2) = \frac{-M\theta}{32\pi} \int_0^\infty ds \rho_2(s) F(M^2, s, q^2). \tag{13}$$

For small momentum transfer ($q^\mu \to 0$)

$$F(M^2, s, q^2) = \frac{1}{M} - \frac{\sqrt{s}}{M^2} + \frac{s}{2M^3} \ln\left(1 + 2\sqrt{M^2/s}\right) + O\left(q^2\right), \tag{14}$$

we get

$$\mu = -\frac{i\theta}{64\pi^2} \int_0^\infty \frac{ds}{\Gamma^2(s) + \left(\frac{\theta}{16}\right)^2 s} \left[\frac{1}{M} - \frac{\sqrt{s - i\epsilon}}{M^2} + \frac{s - i\epsilon}{2M^3} \ln\left(1 + 2\sqrt{\frac{M^2}{s - i\epsilon}}\right)\right]. \tag{15}$$

In the leading order in $1/N$, the eq. (15) is the exact magnetic moment of $n$ fields in the $CP^{N-1}$ model with a Chern-Simons term. Since it is not easy to perform the integration, we consider two extreme cases where $(i)$ the mass of $n$ fields is very large ($M \to \infty$) and $(ii)$ very small ($M \to 0$). In these limits the parity-odd part of Chern-Simons propagator becomes for each cases

$$(i) \qquad G^{a(\text{odd})}_{\lambda\nu}(k) \simeq \tfrac{16}{\theta} \tfrac{\epsilon_{\lambda\rho\nu}k^\rho}{k^2}, \quad (M^2 \gg k^2) \tag{16}$$

$$(ii) \qquad G^{a(\text{odd})}_{\lambda\nu}(k) \simeq \tfrac{16\theta}{\theta^2+1} \tfrac{\epsilon_{\lambda\rho\nu}k^\rho}{k^2}, \quad (M^2 \ll k^2). \tag{17}$$

Repeating calculations with these propagators, we get

$$(i) \qquad \mu = \tfrac{4}{\pi M\theta} \quad (M \to \infty), \tag{18}$$

$$(ii) \qquad \mu = \tfrac{4\theta}{\pi M(\theta^2+1)} \quad (M \to 0). \tag{19}$$

Since the induced spin of $n$ fields is $S = \frac{4}{\pi\theta}$, the gyromagnetic ratio is for each limiting cases

$$(i) \qquad g = 2 \quad (M \to \infty), \tag{20}$$

$$(ii) \qquad g = \tfrac{2\theta^2}{\theta^2+1} \quad (M \to 0). \tag{21}$$

We see here that in the non-relativistic limit (namely in the heavy mass limit, $M \to \infty$) the $g$-factor of anyons is two at the leading order in the $1/N$ expansion.

We now consider the next-to-leading corrections to the magnetic moment when the mass of anyon goes to infinity to see if the $g$-factor of heavy anyon does not get corrections in higher orders in $1/N$. The next-leading order diagrams in $1/N$ for the vertex function are shown in fig.2. Since in the large $M$ limit, the gauge field propagator becomes

$$G_{\mu\nu}(p) = \frac{32}{3\pi\theta^2}\frac{1}{M}\left(\delta_{\mu\nu} - \frac{p_\mu p_\nu}{p^2}\right) + \frac{16}{\theta}\frac{\epsilon_{\mu\rho\nu}p^\rho}{p^2}, \tag{22}$$

the contribution of the parity-even part of the gauge field propagator is suppressed by $1/M$. Therefore, in the large $M$ limit, one may neglect the parity-even part of the gauge field propagator in calculating the vertex function. To get the parity-odd vertex correction, we need to consider the diagrams which has odd number of the gauge propagator. Then, the diagram (a) in fig. 2 is the only diagram that may contribute to the parity-odd part of the vertex function in the next-to-leading order in $1/N$. But, one can see that the diagram does not contribute to the magnetic moment of heavy anyon by the Coleman-Hill argument we used previously. The diagram (a) gives in the large $M$ limit

$$\begin{aligned}\Gamma_\mu^{(2)} = \frac{1}{N}\left(\frac{16}{\theta}\right)^3 \int_{k_1,k_2} & \frac{(2p-k_1)_\delta(2p-2k_1-k_2)_\rho(2p-q-k_1-k_2)_\sigma}{[(p-k_1)^2-M^2]\,[(p-k_1-k_2)^2-M^2]} \\ & \cdot \frac{\epsilon_{\delta\nu\alpha}k_1^\nu}{k_1^2}\frac{\epsilon_{\rho\tau\beta}k_2^\tau}{k_2^2}\frac{\epsilon_{\sigma\kappa\gamma}(q-k_1-k_2)^\kappa}{(q-k_1-k_2)^2}\Gamma_{\mu\alpha\beta\gamma}(q,k_1,k_2;q-k_1-k_2),\end{aligned} \tag{23}$$

where $\Gamma_{\mu\alpha\beta\gamma}(q,k_1,k_2;q-k_1-k_2)$ is the 1-loop 4-photon effective vertex. Because of the gauge invariance and the analyticity of the effective vertex[15],

$$\Gamma_{\mu\alpha\beta\gamma}(q,k_1,k_2;q-k_1-k_2) = q_\mu k_{1\alpha}k_{2\beta}f_\gamma(q,k_1,k_2), \tag{24}$$

where $f_\nu(q, k_1, k_2)$ is a function of $q, k_1, k_2$ and regular at zero momenta. In the vertex function (23), the effecive vertex $\Gamma_{\mu\alpha\beta\gamma}$ is contracted with the parity-odd part of the gauge field propagator, $\Gamma^{(2)}_\mu$ of the diagram (a) vanishes. We therefore find that the next-to-leading order correction in $1/N$ to the magnetic moment of anyon is zero in the heavy anyon limit. Since there is no correction to the Chern-Simons term in the $CP^{N-1}$ model [15, 16, 10], $g = 2$ still holds up to the order of $1/N^2$. Chou *et al.*'s result is therefore verified to this order.

Finally, we consider the two-particle scattering amplitudes of anyons. The two-particle interaction due to the gauge particle exchange is shown in figs. 3a, b. The scattering amplitude for two identical anyons is

$$A_{sym} = A(p_1, p_2; p_1 - q, p_2 + q) + A(p_1, p_2; p_2 + q, p_1 - q), \tag{25}$$

where

$$A(p_1, p_2; p_1 - q, p_2 + q) = -i\Gamma_\mu(p_1, p_1 - q)G_{\mu\nu}(q)\Gamma_\nu(p_2, p_2 + q) \tag{26}$$

with the vertex function

$$\Gamma_\mu(p_1, p_1 - q) = (2p_1 - q)_\mu - i\mu\frac{\epsilon_{\mu\nu\lambda}p_{1\nu}q_\lambda}{M}. \tag{27}$$

For the small momentum transfer, $-q^2 \ll M^2$, we get after simple calculations

$$A_{sym} = 4(p_1 \cdot p_2)\frac{\frac{M}{6\pi}}{\left(\frac{M}{6\pi}\right)^2 + \left(\frac{\theta}{16}\right)^2 q^2}\left[1 + 2\mu M + O(q^2)\right]. \tag{28}$$

Therefore we see that for $2\mu M < -1$ or $-\frac{8}{\pi} < \theta < 0$ two identical anyons attract each other.

In conclusion, we have calculated the magnetic moment of anyons in the $CP^{N-1}$ model with a Chern-Simons term in various limits and found that the gyromagnetic ratio of anyons is 2 up to the next-to-leading order in $1/N$ expansion for the heavy anyon limit. This result is consistent with the result claimed in [6]. We have also

shown that for $-\frac{8}{\pi} < \theta < 0$ two identical anyons attract each other, which indicates that anyon and magnetic flux composit is possible.

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# References

[1] F. Wilczek, Phys. Rev. Lett. **48** (1982) 1144; R. Jackiw and A. N. Redlich, Phys. Rev. Lett. **50** (1982) 555; A. S. Goldhaber, Phys. Rev. Lett. **49** (1982) 905; F. Wilczek and A. Zee, Phys. Rev. Lett. **51** (1983) 2250; Y.-S. Wu, Phys. Rev. Lett. **52** (1984) 2103, and references therein.

[2] R. B. Lauglin, Sience **242** (1988) 525; A. L. Fetter, C. B. Hanna, and R. B. Laughlin, Phys. Rev. **B 39** (1989) 9679; Y.-H Chen, F. Wilczek, E. Witten, and B. Halperin, Int. J. Mod. Phys. **B 3** (1989) 1001.

[3] B. I. Halperin, Phys. Rev. Lett. **52** (1984) 2390; D. Arovas, J. R. Schrieffer, F. Wilczek, Phys. Rev. lett. **53** (1984) 722; B. Blok and X. G. Wen, Phys. Rev. **B 43** (1991) 8337, and references therein.

[4] I.I. Kogan, Phys. Lett. **B262** (1991) 83.

[5] J. K. Jain, Phys. Rev. Lett. **63** (1989) 199.

[6] C. Chou, V.P. Nair, and A.P. Polychronakos, Phys. lett. **B 159** (1993) 105.

[7] I.I. Kogan and G. Semenoff, Nucl. Phys. **B 368** (1992) 718.

[8] G. Gat and R. Ray, Phys. Lett. **B 340** (1994) 162.

[9] S. H. Park, Mod. Phys. Lett. **A 7** (1992) 1579.

[10] G. Ferretti and S. G. Rajeev, Mod. Phys. Lett. **A 7** (1992) 2087.

[11] A. M. Polyakov, Mod. Phys. Lett. **A 3** (1988) 325; I. Dzialoshinsky, A. M. Polyakov, and P. Wiegman, Phys. Lett. **A 127** (1988) 112.

[12] P. B. Wiegman, Phys. Rev. Lett. **60** (1988) 821; 2445 (**E**).

[13] Y. Aharonov and D. Bohm, Phys. Rev. **115** (1959) 485.

[14] I.Ya. Arefyeva and S.I. Azakov, Nucl. Phys. **B 162** (1980) 298.

[15] S. Coleman and B. Hill, Phys. Lett. **B159** (1985) 184.

[16] T. Lee, Phys. Lett. **B171** (1986) 247.

# Figure Captions

**Fig. 1:** The leading order corrections to the vertex function. The solid lines denote fermions, the wavy lines gauge fields, and The dotted lines the auxiliary fields.

**Fig. 2:** The next-to-leading order corrections to the vertex function.

**Fig. 3:** Two-particle interaction due to gauge particle exchange.

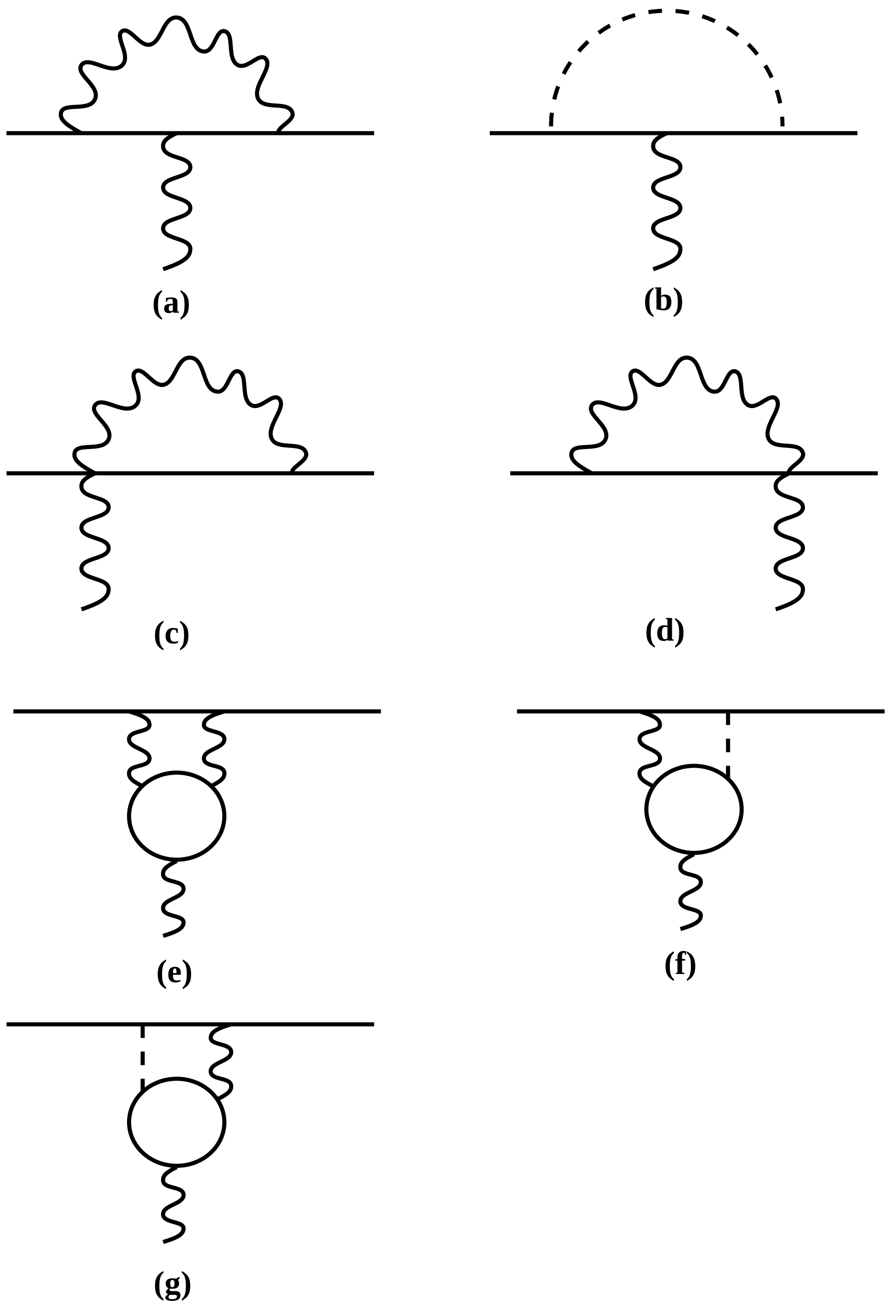



**Figure 1**

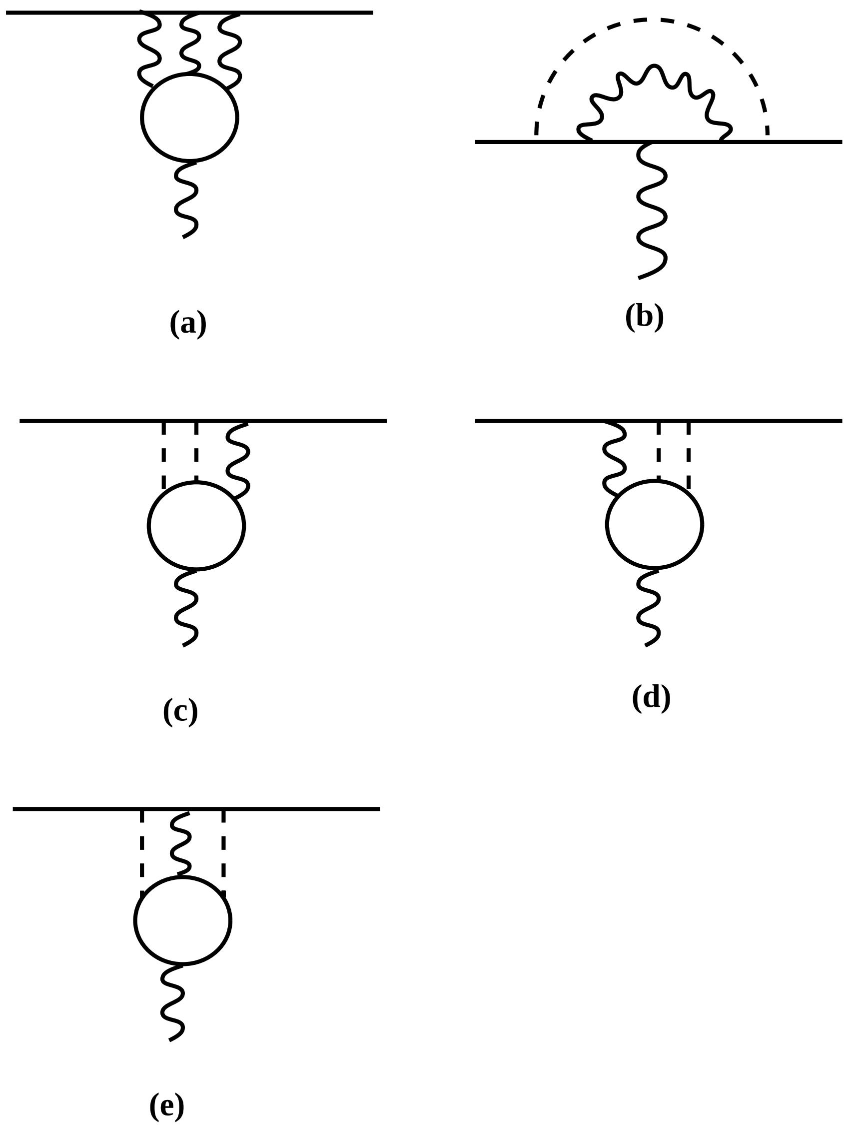
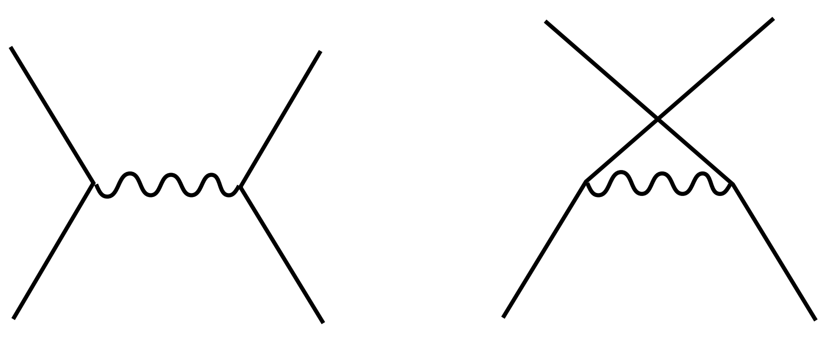



**Figure 2**



**(a)** **(b)**

**Figure 3**